Kerler-Lyubashenko Functors on 4-Dimensional 2-Handlebodies

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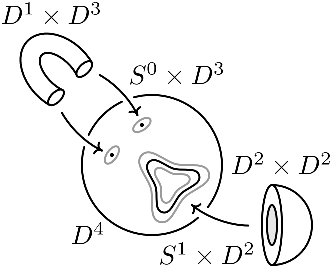
4-Dimensional 2-Handlebodies

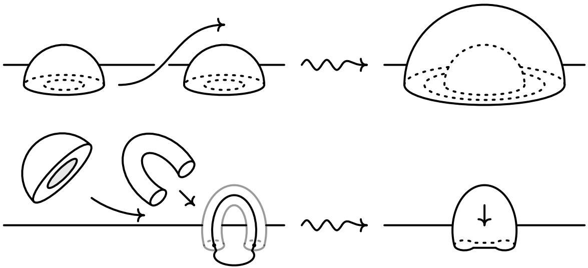
Connected 4-dimensional 2-handlebodies are 4-manifolds obtained from D^4 by attaching a finite number of handles of index 1 and 2



Handle Moves

Every diffeomorphism is implemented by a finite sequence of

- isotopies of attaching maps
- handle slides
- creation/removal of canceling pairs of handles of index $k/k+1$



2-Deformations and 2-Equivalence

Diffeomorphisms that do not create/remove canceling pairs of handles of index $2/3$ and $3/4$ are called 2-deformations

Induced equivalence relation: 2-equivalence

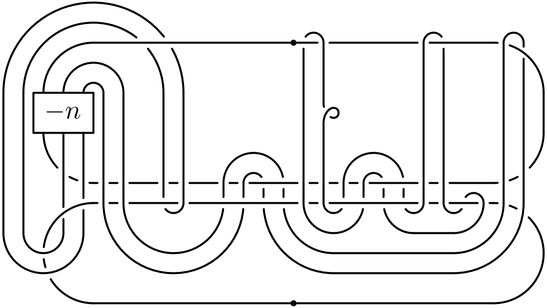
Question: Is every diffeomorphism between 4-dimensional 2-handlebodies a 2-deformation?

Answer widely believed to be negative

Andrews-Curtis Conjecture

Conjecture (Andrews-Curtis): Every contractible connected 4-dimensional 2-handlebody is 2-equivalent to D^4

Potential counterexamples: Akbulut-Kirby 4-balls

$\Delta_n =$ 

Category of 4-Dimensional 2-Handlebodies

$4\mathrm{HB}$ braided monoidal category with

- objects: connected 3-dimensional 1-handlebodies
- morphisms: 2-equivalence classes of connected 4-dimensional 2-handlebodies
- tensor product: boundary connected sum
- braiding: 1-handle permutation

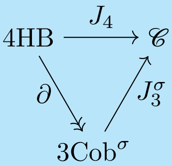
Main Result

Theorem (Beliakova-D)

If $\mathscr{C}$ is a unimodular ribbon category, there exists a braided monoidal functor

$$J_4 : 4\mathrm{HB} \to \mathscr{C}$$

sending the solid torus to $\int_{X \in \mathscr{C}} X \otimes X^*$, and if $\mathscr{C}$ is also factorizable, then



for the category $3\mathrm{Cob}^\sigma$ of connected framed relative 3-dimensional cobordisms

Instability of 2-Exotic Pairs

Diffeomorphic 4-dimensional 2-handlebodies that are not 2-equivalent form 2-exotic pairs

Diffeomorphic 4-dimensional 2-handlebodies become 2-equivalent after a finite number of boundary connected sums with $S^2 \times D^2$

Any invariant that is multiplicative under boundary connected sum cannot detect 2-exotic pairs unless it vanishes against $S^2 \times D^2$

Detection of 2-Exotic Pairs

Proposition (Beliakova-D)

When $\mathscr{C} = H\text{-mod}$ *then*

$$J_4(S^1 \times D^3) = \varepsilon(\Lambda) \qquad J_4(S^2 \times D^2) = \lambda(1)$$

for a two-sided cointegral $\Lambda \in H$ *and a left integral* $\lambda \in H^*$*, so*

- $J_4(S^1 \times D^3) \neq 0 \Leftrightarrow H$ *semisimple*
- $J_4(S^2 \times D^2) \neq 0 \Leftrightarrow H$ *cosemisimple (i.e.* H^* *semisimple)*

Example: $u_q\mathfrak{sl}_2$ at $q = e^{\frac{2\pi i}{r}}$ with $3 \leqslant r \in \mathbb{Z}$

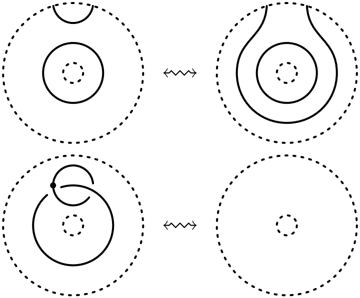
- ribbon and unimodular
- neither semisimple nor cosemisimple
- factorizable $\Leftrightarrow r \not\equiv 0 \mod 4$

Kirby Tangles and Kirby Moves

Kirby tangles are framed tangles with components of three kinds

- dotted unknots with framing 0
- undotted knots with arbitrary framing
- undotted arcs joining consecutive boundary vertices

Kirby moves are

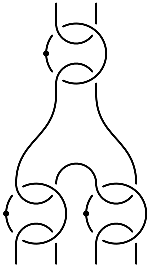


Category of Kirby Tangles

KTan category with

- objects: natural numbers
- morphisms: Kirby tangles up to Kirby moves

Example:



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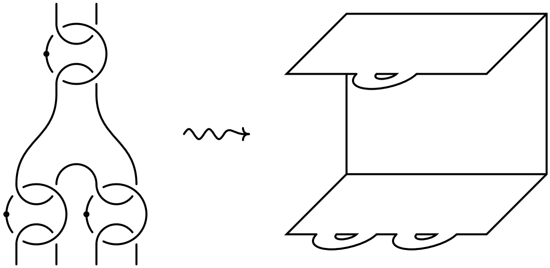

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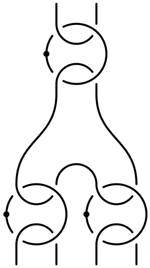
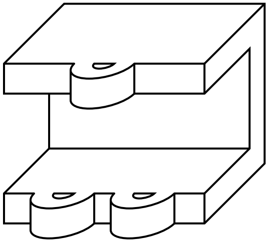


$\rightsquigarrow$



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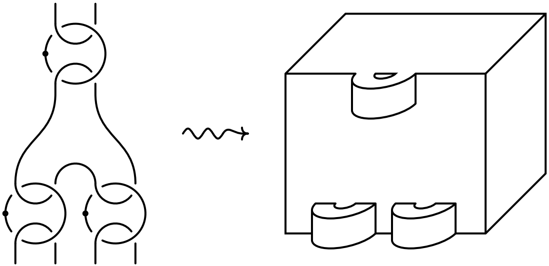

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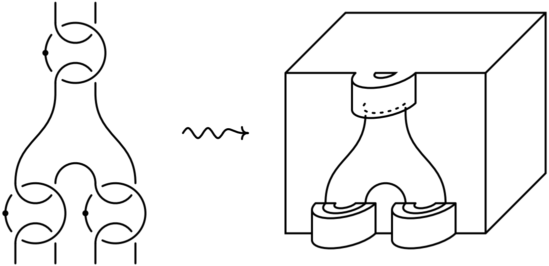
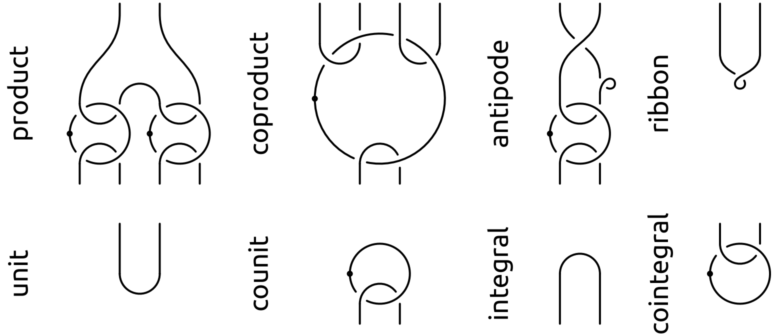


Preview: Algebraic Presentation of $4\mathrm{HB}$

Theorem (Bobtcheva-Piergallini)

$4\mathrm{HB} \cong \mathrm{KTan}$ *is the free braided monoidal category generated by a pre-modular Hopf algebra*



Unimodular Ribbon Hopf Algebras

A Hopf algebra is an algebra H over $\Bbbk$ with

- a coproduct $\Delta(x) = x_{(1)} \otimes x_{(2)} \in H \otimes H$ for every $x \in H$
- a counit $\varepsilon(x) \in \Bbbk$ for every $x \in H$
- an antipode $S(x) \in H$ for every $x \in H$

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- an antipode $S(x) \in H$ for every $x \in H$
- an R-matrix $R = R'_i \otimes R''_i \in H \otimes H$
- a pivotal element $g \in H$

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- an R-matrix $R = R'_i \otimes R''_i \in H \otimes H$
- a pivotal element $g \in H$
- a two-sided cointegral $\Lambda \in H$
- a left integral $\lambda \in H^*$

Example: Hopf Algebra Structure on $u_q\mathfrak{sl}_2$

- $q = e^{\frac{2\pi i}{r}}$ with $3 \leqslant r \in \mathbb{Z}$
- $r' = \frac{r}{\gcd(r,2)}$
- $u_q\mathfrak{sl}_2$ algebra over $\mathbb{C}$ with
  - generators: E, F, K
  - relations: $E^{r'} = F^{r'} = 0, \quad K^{r'} = 1,$
    $KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E,F] = \dfrac{K - K^{-1}}{q - q^{-1}}$
- Hopf algebra structure:

$$\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E & \varepsilon(E) &= 0 & S(E) &= -EK^{-1} \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1 & \varepsilon(F) &= 0 & S(F) &= -KF \\ \Delta(K) &= K \otimes K & \varepsilon(K) &= 1 & S(K) &= K^{-1} \end{aligned}$$

- $\mathscr{C} = u_q\mathfrak{sl}_2\text{-mod}$ is a rigid monoidal category

Example: Ribbon Structure on $u_q\mathfrak{sl}_2$

- $\{a\} := q^a - q^{-a}, \quad [a] := \frac{\{a\}}{\{1\}}, \quad [a]! := [a][a-1]\cdots[1] \quad \forall\, a \in \mathbb{N}$
- R-matrix:

$$R = \frac{1}{r'} \sum_{a,b,c=0}^{r'-1} \frac{\{1\}^a}{[a]!} q^{\frac{a(a-1)}{2} - 2bc} K^b E^a \otimes K^c F^a$$

- pivotal element:

$$g = K$$

- $\mathscr{C} = u_q\mathfrak{sl}_2\text{-mod}$ is a ribbon category

Example: Integral and cointegral of $u_q\mathfrak{sl}_2$

- $r'' = \frac{r}{\gcd(r,4)}$
- two-sided cointegral

$$\Lambda = \frac{\{1\}^{r'-1}}{\sqrt{r''}[r'-1]!} \sum_{a=0}^{r'-1} E^{r'-1} F^{r'-1} K^a$$

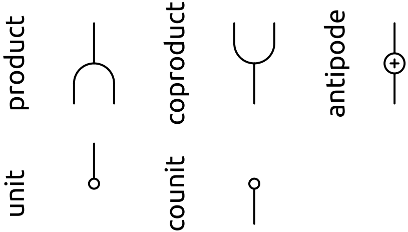
- $\{E^a F^b K^c \mid 0 \leqslant a, b, c \leqslant r' - 1\}$ basis of $u_q\mathfrak{sl}_2$
- left integral

$$\lambda\left(E^a F^b K^c\right) = \frac{\sqrt{r''}[r'-1]!}{\{1\}^{r'-1}} \delta_{a,r'-1} \delta_{b,r'-1} \delta_{c,r'-1}$$

- $\mathscr{C} = u_q\mathfrak{sl}_2\text{-mod}$ is a unimodular ribbon category

Pre-Modular Hopf Algebras

A braided Hopf algebra in a braided monoidal category $\mathscr{C}$ is an object $\mathscr{H} \in \mathscr{C}$ equipped with



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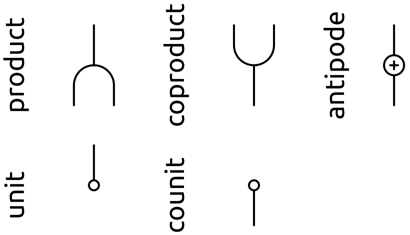

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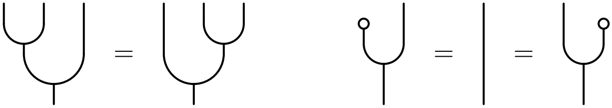

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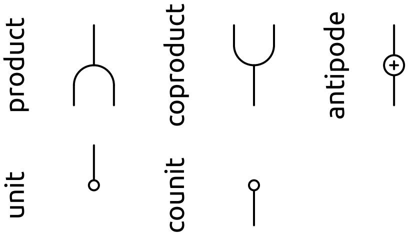

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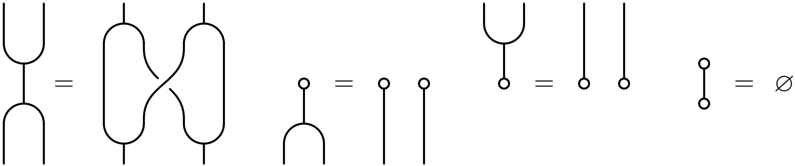

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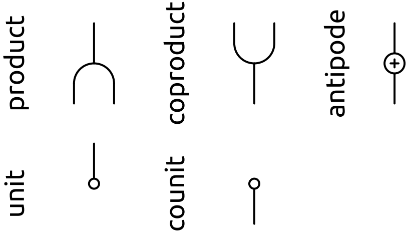

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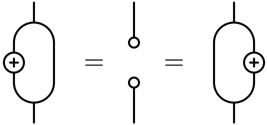

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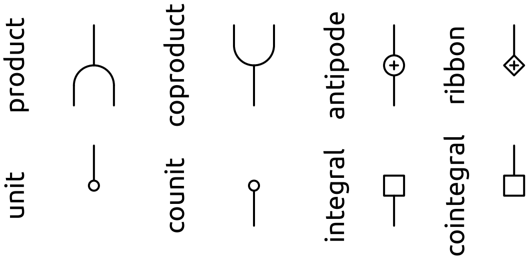

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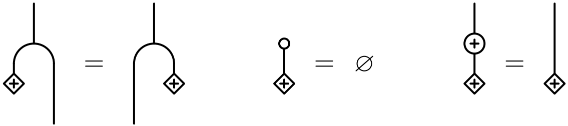

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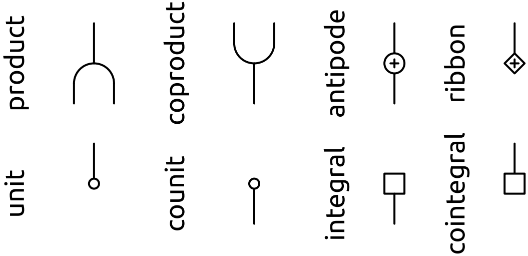

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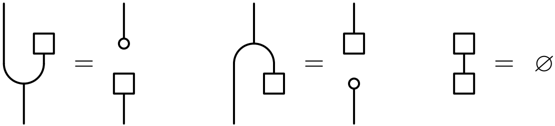

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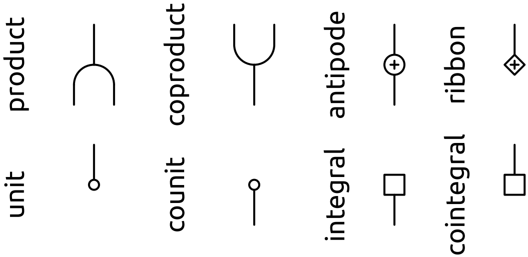

satisfying

and more...

Transmutation

- H unimodular ribbon Hopf algebra
- H is not in general a pre-modular Hopf algebra in $\mathrm{Vect}_{\Bbbk}$
- ad adjoint representation
  - vector space H
  - adjoint H-action given by

$$x \triangleright y := x_{(1)} y S(x_{(2)})$$

    for all $x, y \in H$
- ad is a pre-modular Hopf algebra in $H\text{-}\mathrm{mod}$

Algebraic Presentation of $4\mathrm{HB}$

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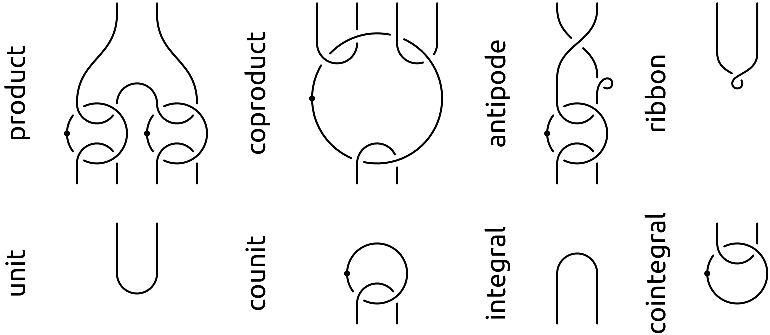
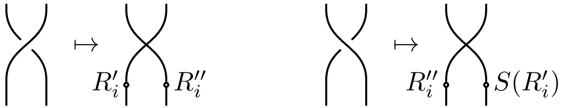


Algorithm

H unimodular ribbon Hopf algebra, T closed Kirby tangle

- insert R beads at crossings



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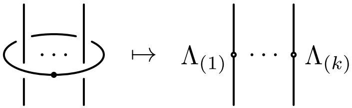
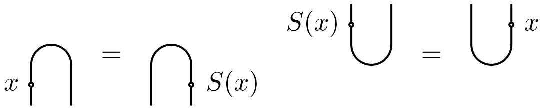
- insert R beads at crossings
- insert Λ beads at dotted components



Algorithm

H unimodular ribbon Hopf algebra, T closed Kirby tangle

- insert R beads at crossings
- insert Λ beads at dotted components
- collect beads in one place



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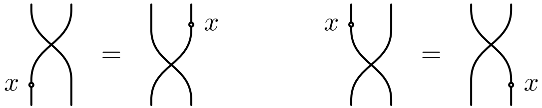

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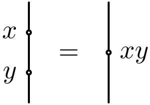
- insert R beads at crossings
- insert Λ beads at dotted components
- collect beads in one place
- multiply beads together



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$$J_4(T) = \prod_{i=1}^{k} \lambda(x_i g^{d_i - 1})$$

where d_i is the Whitney degree of the ith component

Computations

Proposition (Beliakova-D)

If $\mathscr{C} = u_q\mathfrak{sl}_2\text{-mod}$ *at* $q = e^{\frac{2\pi i}{r}}$ *with* $3 \leqslant r \in \mathbb{Z}$ *then*

$$J_4(\mathbb{C}P^2 \smallsetminus \mathring{D}^4) = \begin{cases} i^{\frac{r-1}{2}} q^{\frac{r+3}{2}} & \textit{if } r \equiv 1 \mod 2 \\ \left(\frac{2}{r'}\right) i^{\frac{r'-1}{2}} q^{\frac{r'+3}{2}} & \textit{if } r \equiv 2 \mod 4 \\ -q^{\frac{r''+3}{2}} & \textit{if } r \equiv 4 \mod 8 \\ 0 & \textit{if } r \equiv 0 \mod 8 \end{cases}$$

$$J_4((S^2 \times S^2) \smallsetminus \mathring{D}^4) = (-1)^{r'-1}$$